**Math 10B with Professor Stankova**
**Worksheet, Discussion #18; Tuesday, 4/2/2019**
**GSI name: Roy Zhao**

# Median and Mean

## Concepts

1. The **mean** of a continuous random variable is the same as the expected value and is given by

$$\mu = E[X] = \int_{-\infty}^{\infty} x f(x) dx.$$

A CDF is a function $F(x)$ where $F(x) = P(X \leq x)$, it tells us that probability of getting a value less than or equal to $x$. It is just defined as $F(x) = \displaystyle\int_{-\infty}^{x} f(x) dx$. It satisfies three important properties:

- $F(x)$ is nondecreasing. So if $x \leq y$, then $F(x) \leq F(y)$.
- $\lim\limits_{x \to -\infty} F(x) = 0$.
- $\lim\limits_{x \to \infty} F(x) = 1$.

The **median** is the point that is at the midpoint of the probability distribution. It is when $P(X \leq x) = 0.5$ or when the CDF is equal to 0.5.

## Example

2. Let $g(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2 - x & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$. Find $c$ such that $f(x) = cg(x)$ is a PDF. Graph $f$ and the CDF $F$. Find the mean and median of $f(x)$.

**Solution:** Drawing $g$, we see that it is a triangle of height 1 and base 2 and so the area under $g$ is 1 so $c = 1$. Then, the CDF $F$ is given by

$$F(x) = \begin{cases} 0 & x \leq 0 \\ \int_0^x t dt & 0 \leq x \leq 1 \\ \int_0^1 t dt + \int_1^x (2-t) dt & 1 \leq x \leq 2 \\ \int_0^1 t dt + \int_1^2 (2-t) dt & 1 \leq x \end{cases} = \begin{cases} 0 & x \leq 0 \\ x^2/2 & 0 \leq x \leq 1 \\ \frac{-x^2+4x-2}{2} & 1 \leq x \leq 2 \\ 1 & 1 \leq x \end{cases}$$

The median is when $F(x) = \frac{1}{2}$ and setting them equal gives $x^2/2 = 1/2$ or $x = 1$. The mean is

$$\mu = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 x^2dx + \int_1^2 x(2-x)dx = \frac{1}{3} + \frac{2}{3} = 1.$$

In this case, our mean and median are the same.

## Problems

3. **TRUE** False It is possible for the mean for a discrete PDF to not exist.

**Solution:** Consider the distribution such that choosing $2^n$ has probability $\frac{1}{2^n}$. Then the mean doesn't exist, but this is a discrete PDF.

4. **TRUE** False Another name for the mean of a PDF is the expected value.

5. True **FALSE** For a discrete PDF, the mean occurs with nonzero probability.

**Solution:** Consider rolling a dice. The mean is 3.5, which we cannot roll.

6. True **FALSE** There exists a uniform distribution on all the real numbers.

**Solution:** We cannot make the total area 1.

7. Let $g(x) = \begin{cases} x^2 & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$. Find $c$ such that $f(x) = cg(x)$ is a PDF. Graph $f$ and the CDF $F$. Find the mean and median of $f(x)$.

**Solution:** First we calculate

$$\int_{-\infty}^{\infty} g(x)dx = \int_{-1}^{1} x^2dx = \frac{2}{3}.$$

Therefore, we must have that $\frac{2}{3}c = 1$ or $c = \frac{3}{2}$. The CDF is

$$F(x) = \begin{cases} 0 & x \leq -1 \\ \int_{-1}^{x} 3/2t^2 dt & -1 \leq x \leq 1 \\ \int_{-1}^{1} 3/2t^2 dt & 1 \leq x \end{cases} = \begin{cases} 0 & x \leq -1 \\ \frac{x^3+1}{2} & -1 \leq x \leq 1 \\ 1 & 1 \leq x \end{cases}.$$

The median is when $F(x) = 1/2$ or when $x^3 + 1 = 1$ which is $x = 0$. The mean is

$$\mu = \int_{-\infty}^{\infty} x f(x) dx = \int_{-1}^{1} 3/2x^3 dx = 0,$$

since $3/2x^3$ is an odd function. So again, the median and mean align.

8. Let $F(x) = \frac{x-1}{x+1}$ for $x \geq 1$ and 0 for $x \leq 1$. Show that $F$ is a CDF. Find the PDF associated with it and the probability that we choose a number between 1 and 2.

**Solution:** This is a CDF because it is continuous since $F(1) = 0$ and $\lim_{x\to\infty} F(x) = 1$ and $F$ is non-decreasing. The PDF is

$$f(x) = \frac{d}{dx}F(x) = \frac{2}{(x+1)^2}$$

for $x \geq 1$ and 0 for $x \leq 1$. The probability that we choose a number between 1 and 2 is

$$\int_{1}^{2} f(x)dx = F(2) - F(1) = \frac{1}{3}.$$

9. Let $g(x) = \begin{cases} e^{-x} & -1 \leq x \\ 0 & \text{otherwise} \end{cases}$. Find $c$ such that $f(x) = cg(x)$ is a PDF. Graph $f$ and the CDF $F$. Find the mean and median of $f(x)$.

**Solution:** First we calculate

$$\int_{-\infty}^{\infty} g(x)dx = \int_{-1}^{\infty} e^{-x}dx = -e^{-x}|_{-1}^{\infty} = e.$$

So $ec = 1$ and $c = 1/e$. The CDF is

$$F(x) = \begin{cases} 0 & x \leq -1 \\ \int_{-1}^{x} 1/ee^{-t}dt & x \geq -1 \end{cases} = \begin{cases} 0 & x \leq -1 \\ 1 - e^{-x}/e & x \geq -1 \end{cases}.$$

The median is when $F(x) = 1/2$ or when $1 - e^{-x}/e = \frac{1}{2}$ which is $x = \ln 2 - 1$. The mean is

$$\mu = \int_{-\infty}^{\infty} xf(x)dx = \int_{-1}^{\infty} xe^{-x}/edx = (-xe^{-x} - e^{-x})|_{-1}^{\infty}/e = (e-e)/e = 0.$$

10. Let $g(x) = \begin{cases} \frac{1}{x^4} & x \leq -1 \\ 0 & \text{otherwise} \end{cases}$. Find $c$ such that $f(x) = cg(x)$ is a PDF. Graph $f$ and the CDF $F$. Find the mean and median of $f(x)$.

**Solution:** First we calculate

$$\int_{-\infty}^{\infty} g(x)dx = \int_{-\infty}^{-1} \frac{1}{x^4}dx = \frac{-1}{3x^{-3}}|_{-\infty}^{-1} = \frac{1}{3}.$$

Therefore, $c(1/3) = 1$ so $c = 3$. The CDF is

$$F(x) = \begin{cases} \int_{-\infty}^{x} \frac{3}{t^4}dt & x \leq -1 \\ 1 & x \geq -1 \end{cases} = \begin{cases} \frac{-1}{x^3} & x \leq -1 \\ 0 & x \geq -1 \end{cases}.$$

The median is when $F(x) = 1/2$ or when $-1/x^3 = 1/2$ which is $x = -\sqrt[3]{2}$. The mean is

$$\mu = \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{-1} \frac{3}{x^3}dx = \frac{-3}{2}.$$

11. Let $g(x) = \begin{cases} \frac{1}{x^4} & 2 \leq x \\ 0 & \text{otherwise} \end{cases}$. Find $c$ such that $f(x) = cg(x)$ is a PDF. Graph $f$ and the CDF $F$. Find the mean and median of $f(x)$.

**Solution:** First we calculate

$$\int_{-\infty}^{\infty} g(x)dx = \int_{2}^{\infty} 1/x^4dx = -1/(3x^3)|_2^{\infty} = 1/24.$$

Therefore, $c/24 = 1$ and $c = 24$. The CDF is

$$F(x) = \begin{cases} 0 & x \leq 2 \\ \int_2^x 24/t^4dt & t \geq 2 \end{cases} = \begin{cases} 0 & x \leq 2 \\ 1 - 8/x^3 & x \geq 2 \end{cases}.$$

The median is when $F(x) = 1/2$ or when $1 - 8/x^3 = \frac{1}{2}$ which is $x = \sqrt[3]{16}$. The mean is

$$\mu = \int_{-\infty}^{\infty} xf(x)dx = \int_{2}^{\infty} 24/x^3dx = -12/x^2|_2^{\infty} = 3.$$